Calibrating Risk Aversion in Additive Multivariate Utility Functions

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ABSTRACT

Additive multivariate utility functions are common in applications of economic decision-making. They exist in many areas of multi-attribute decisions and feature prominently in several behavioral economic decision models. For predictions or welfare analyses using such models, it is often necessary to calibrate both ordinal and cardinal preferences in them. One aspect of cardinal preferences is risk aversion. However, the concept of risk aversion in additive multivariate utility functions is poorly understood. In fact, it is impossible to compare two additive multivariate utility functions solely with respect to their risk aversion regarding one or more attributes - changing their risk aversion changes ordinal preferences. We introduce the class of contextual additive multivariate utility functions and consider a subclass of increases in Arrow-Pratt risk aversion, namely those that increase risk aversion in the sense of Ross. In this setting we show that risk premiums change monotonically in risk aversion regarding a single attribute which eases the process of calibrating preference functionals. Additionally, ordinal preferences change in a sensible manner when Ross risk aversion is increased. We apply our procedure to calibration and risk premiums in the Kőszegi-Rabin decision model.

Keywords: Multivariate Utility Functions, Risk Aversion, Preference Calibration, Behavioral Economics
JEL: D01, D81, D90

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1 Introduction

To describe behavior in decisions under risk, economic theory often considers models in which decision-makers are concerned with more than one attribute. Examples include applied fields in which arguments other than wealth are of importance, such as health economics (Viscusi and Evans, 1990) and labor economics (Chetty, 2006). Moreover, theories of multiple attributes are at the heart of several modern developments in behavioral economics. Models such as regret theory (Loomes and Sugden, 1982), fairness preferences (Fehr and Schmidt, 1999) or reference-dependent preferences (Köszegi and Rabin, 2007) trade off wealth consequences with other stimuli such as the regret from foregone alternatives, the relative positioning in a group, or divergence from a reference point. In almost all of these applications, an additive multivariate utility (AMU) function is the predominant parametric assumption (e.g., Hall and Jones, 2007; Einav et al., 2010).\(^1\) Particularly in behavioral economics, additive forms are commonly assumed to make the representation of preferences as simple as possible (see, e.g., the models referenced above).

To predict decision-making behavior or pursue welfare analysis, knowledge of the decision-makers’ preferences is often necessary. For example, to model insurance demand using regret theory, we need to know how much potential regret decision-makers are willing to accept in order to reduce the riskiness of their long term wealth (Braun and Muermann, 2004). Similarly, for analyzing the welfare distortions due to asymmetric information in the annuity market, we need to know how consumers trade off risk in lifetime consumption with their preferences to bequeath (Einav et al., 2010). The literature has thus spent continued effort on calibrating preference functionals from both experimental data (Andersen et al., 2008; Bleichrodt et al., 2010; Attema et al., 2016) and naturally occurring data (Chetty, 2006; Einav et al., 2010; Barseghyan et al., 2013). Calibration in multivariate utility functions is not limited to parameters that capture ordinal preferences across attributes, but also extends to parameters characterizing cardinal preference information. Specifically, this entails knowledge of risk attitude, which in additive models is captured in the component functions of the individual attributes. Since calibration is the inference of preferences from behavior, it is thus of interest to know, how changes in risk attitude regarding one of the attributes impact decision-makers’ behavior. In the univariate setting, risk attitude is captured by the index

\(^1\)See Pollak (1967) for necessary and sufficient conditions justifying such assumptions.
developed in Arrow (1970) and Pratt (1964). This has naturally led to the use of the Arrow-Pratt index to measure single-attribute risk attitude in AMU functions (Keeney, 1973; Chetty, 2006). However, the concept of risk attitude in multivariate utility functions, even in the simple case of AMU functions, remains poorly understood (Epstein and Zin, 1989; Bommier et al., 2012).

In this paper, we first remark that changing risk attitude in AMU functions without leaving the additive paradigm, always changes ordinal preferences. Therefore, the Arrow-Pratt coefficient of risk aversion regarding a single attribute has no general interpretation in AMU functions. We then consider different concepts of single-attribute risk premiums in the multivariate setting. We derive conditions under which changes in single-attribute risk attitude lead to monotonic changes in these risk premiums. For this, we introduce a new class of utility functions, the contextual additive multivariate utility (CAMU) functions which are based on the concept of contextual utility initially introduced by Wilcox (2011). For CAMU functions, we demonstrate that a restricted class of risk-aversion-increasing Arrow-Pratt transformations, namely those defined by Ross (1981), leads to monotonic changes of risk premiums in all attributes. Although these transformations still change ordinal preferences by necessity, we show these changes to be in accordance with intuition. We demonstrate the usefulness of our results in an applications. We show how they lead to monotone risk premiums in the K˝oszegi and Rabin (2007) decision model which eases calibration of the model.

To exemplify the problem analyzed in this paper, consider the decision situation displayed in Figure 1. The decision-maker is faced with the decision between lotteries $L_1$ and $L_2$. $L_1$ is risky in terms of the first attribute, which we will call wealth, $w$. It does, however, offer a higher expected value of $w$ than $L_2$ which is riskless with regard to wealth. $L_1$ additionally has a lower second attribute, which we will call health, $h$. We now assume the expected utility maximizing decision-maker to evaluate the lotteries with utility function $U(w, h) = 1 - e^{-\xi w} + h^{0.5}$. In the univariate case, $\xi$ simply denotes the risk aversion with regard to $w$. Here, however, the relationship is not as clear. This becomes apparent when considering that for $\xi = 0.0002$, the decision-maker prefers $L_2$ (the riskless lottery) to $L_1$ (the risky lottery). If we now increase $\xi$ to 0.0004, these preferences reverse. A higher coefficient of absolute risk aversion in this CARA utility function over wealth actually leads to the selection of the more risky lottery.

The intuition behind the puzzling behavior in the example and the result of this paper in general is that an increase in risk aversion in the univariate case is dependent on the condition that
univariate utility functions are unique up to a positive affine transformation. In the multivariate case, however, applying a positive affine transformation to a component of the multivariate utility function changes the ordinal preferences of the decision-maker which makes the utility functions generally incomparable with regard to risk aversion (Kihlstrom and Mirman, 1974). In the example above, changing $\xi$ not only changes the curvature with respect to wealth, but also increases the weight of wealth relative to health in the preference functional.\(^2\)

The study of risk aversion originates in the univariate case, usually referred to as the utility function over money. Arrow (1970) and Pratt (1964) establish a series of results on risk aversion in this context. They show that the coefficient of risk aversion can be related to the risk premium and that an increasing and concave transformation of the utility function increases the risk aversion coefficient. Due to the particularities of the univariate case, changing the risk aversion of a preference functional preserves the ordinal preferences of the decision-maker. Based on this work, Ross (1981) introduce a stronger measure of risk aversion. As is for example shown in Eeckhoudt et al. (1996), the resulting stronger ordering of utility functions allows for deriving unambiguous comparative statics in more situations than the more general measure of Arrow (1970) and Pratt (1964). Diamond and Stiglitz (1974) and Kihlstrom and Mirman (1974, 1981) consider risk aversion in multivariate utility functions. Kihlstrom and Mirman (1974) make a convincing argument why such general analyses can only cover variations in the utility function which preserve the ordinal preferences over the outcome space. Prior analyses of risk aversion in multivariate utility functions have thus been limited to transformations of the preference functional which preserve the original ordinal preferences (see, e.g., Bommier et al., 2012).\(^3\) This, however, limits the comparable functions to relatively few cases and makes all AMU functions incomparable (see our Proposition 1).

\(^2\)Note that a similar counterexample could be constructed if the increase in risk aversion would decrease the relative weight of wealth, or if the utility from health was linear.

\(^3\)There is another stream of literature which is closely connected to the issue at hand. Karni (1983a,b) considers state dependent preferences with similar techniques and results as have been applied to multivariate risk aversion. He limits his analyses to “comparable” preferences which are analogous to preserved ordinal preferences.
The topic is also related to intertemporal utility theory. Additive intertemporal utility models can be seen as a special case of AMU functions in the sense that in most applications the model is a weighted combination of the same utility function applied to different arguments. In the literature, it is well known that simply applying a Pratt transformation to this utility function does not only make the decision-maker more risk averse but that the intertemporal rate of substitution is also affected (Richard, 1975; Bommier, 2007). As with other problems of risk aversion in multivariate utility, the common way to address the issue is to leave the additive paradigm and consider only transformations which do not affect the ordinal preferences (e.g., Epstein and Zin, 1989).

The literature does not yet offer a way to maintain the additive structure of the multivariate utility function and store meaning on the risk aversion for a single attribute. The only exception to this is a comparative static result derived by Jindapon and Neilson (2007). They use a transformation which fixes marginal utility at the expectation of a risk to achieve monotonicity properties comparable to those derived for our model. As is discussed in Section 4, their transformation, which was derived for a different purpose, is not as flexible as the contextual utility functions utilized here. In an extension of the result by Jindapon and Neilson (2007), Liu and Wang (2017) recently developed a comparative static result in an additive setting using contextual utility. In the same setting as Jindapon and Neilson (2007) they showed a result related to our Theorem 1 regarding n-th degree changes in risk. Our result builds on theirs but is different in the sense that they analyze a maximization problem, which cannot directly inform about risk premiums. Also, while their result is considering generalizations of risk aversion concepts, we are interested in the possible applications for calibrating decision models.

Our analysis is necessary, because the common approach of applying a concave transformation to the entire preference functional (which is equivalent to leaving the additive paradigm) cannot provide insight into certain matters. Particularly the calibration of behavioral decision models with an additive structure is not covered by it.

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4Optimal prevention as studied in Liu and Wang (2017) is a maximization problem, while the level of the risk premium is a problem of willingness to pay. The two concepts are related, but the answer to one of them does not necessarily imply the answer to the other. See, e.g., Jaspersen (2016, footnote 6) for a numerical example in which the two concepts lead to different answers.
2 Theoretical result

2.1 Notation

Consider a utility function in two attributes \( x_1 \in X_1 \) and \( x_2 \in X_2 \) such that \( U(x_1, x_2) : X_1 \times X_2 \mapsto \mathbb{R} \). Here, we will only consider functions which are twice continuously differentiable in both attributes. Throughout the paper, we will denote the attribute of interest as \( i \) and the other attribute as \(-i\). A prospect in attribute \( i \) is risky when denoted by a tilde (as in \( \tilde{x}_i \)). The coefficient of absolute risk aversion of utility function \( U(\cdot) \) regarding attribute \( i \) is defined as (Keeney, 1973; Chetty, 2006)

\[
    r^U_i(x_i, x_{-i}) = -\frac{U_{ii}(x_i, x_{-i})}{U_i(x_i, x_{-i})},
\]

whereas \( U^i \) and \( U^{ii} \) denote the first and second partial derivative of \( U(\cdot) \) towards attribute \( i \), respectively. We will focus on cases in which \( U^i > 0, U^{ii} < 0 \forall i \). Keeney (1973) defines the class of risk independent utility functions as those in which \( r_i \) is independent of \( x_{-i} \) for all values of \( x_i \). In such functions, the risk aversion coefficient regarding attribute \( i \) at some value \( x_i \) is thus the same for all possible values of the other attribute \(-i\). For two attributes, he shows these functions to be of the form

\[
    U(x_1, x_2) = u_1(x_1) + u_2(x_2) + ku_1(x_1)u_2(x_2).
\]

In this form, we will call the functions \( u_i(x_i) \) component utility functions for attribute \( i \). \( k \) is a constant to be empirically evaluated. The class of additive multivariate utility (AMU) functions is the class of all risk independent utility functions with \( k = 0 \).

In the univariate case, a utility function \( v(x) \) is considered strictly more Arrow-Pratt risk averse than a utility function \( u(x) \) if \( r^v(x) > r^u(x) \forall x \). Pratt (1964) shows that for each utility function \( v(x) \) which is strictly more Arrow-Pratt risk averse than \( u(x) \) for all values of \( x \), there is a function \( g(\cdot) \) with \( g' > 0 \) and \( g'' < 0 \) such that \( v(x) = g(u(x)) \). Ross (1981) defines a stronger comparison of two utility functions. Here, a utility function \( v(x) \) is more Ross risk averse than \( u(x) \) for all \( x^a, x^b \in [x_{\min}; x_{\max}] \) if there exists a \( \lambda > 0 \) such that

\[
    \frac{v''(x^a)}{u''(x^a)} \geq \lambda \geq \frac{v'(x^b)}{u'(x^b)}.
\]
From this definition, it is obvious that more Ross risk averse implies more Arrow-Pratt risk averse. The opposite, however, is not true. Ross initially developed his concept of risk aversion to obtain unambiguous comparative statics in which increases in Arrow-Pratt risk aversion could not render such results. His examples included a risk-return trade-off if the initial wealth of the decision-maker is random and a standard portfolio problem with two risky goods. Since then, Ross risk aversion has been applied to other situations such as risk taking behavior in the presence of background risk (Eeckhoudt et al., 1996) and prevention (Jindapon and Neilson, 2007).

Similar to Pratt’s (1964) result on Arrow-Pratt risk aversion increasing transformations, Ross also introduces a transformation that is necessary and sufficient for increasing Ross risk aversion. He shows that for every pair of increasing utility functions \( u(x) \) and \( v(x) \) with \( v(x) \) being more Ross risk averse than \( u(x) \) as described in equation (3), there exists a function \( h(x) > 0 \) with

\[
-\lambda u'(x) < h'(x) \leq 0 \quad \text{and} \quad h''(x) \leq 0
\]

such that \( v(x) = \lambda u(x) + h(x) \). Note that other than

\[
-\lambda u'(x) < h'(x),
\]

there is no restriction on the scaling of \( v(x) \). If we rewrite equation (3) simply as

\[
\frac{v''(x^a)}{v'(x^b)} \geq \frac{-u''(x^a)}{u'(x^b)},
\]

we can see that if \( v(x) \) is more Ross risk averse than \( u(x) \), the same will be true for \( bv(x) \) for all \( b > 0 \).

Pratt (1964) introduces the concept of the risk premium, \( \pi \). This is the amount of money an individual would be willing to pay in order to receive the expected value of a risky prospect instead of the risky prospect. In the univariate case this corresponds to:

\[
E[u(\bar{x})] = u(E[\bar{x}] - \pi).
\]

In the multivariate case, one can imagine a similar definition of the risk premium. There is, however, the question of which attribute pays this risk premium. We thus define the risk premium using two indices.\(^5\) The first index denotes the attribute in which the premium is paid. For this payment, the decision-maker exchanges the risky prospect of the attribute denoted by the second index for its actuarial equivalent. Letting the attribute for which the actuarial equivalent is gained be \( x_i \), we can define the on-attribute risk premium, \( \pi_{i,i} \) via

\[
E[U(\tilde{x}_i, \tilde{x}_{-i})] = E[U(E[\tilde{x}_i] - \pi_{i,i}, \tilde{x}_{-i})]. \tag{4}
\]

\(^5\)Paroush (1975) offers a broader definition of the risk premium in the multivariate case than is used here. He relates it to the coefficient of risk aversion in multivariate utility functions with equal ordinal preferences. Since ordinal preferences change between different AMU functions, the results derived here do not hold under his more general definition as is discussed in Appendix A.
We will also be concerned with the cross-attribute risk premium. $\pi_{-i,i}$ describes the amount of attribute $x_{-i}$ the individual is willing to give up in order to receive the expected value of the risky prospect $\tilde{x}_i$ instead of the risky prospect itself.

$$E[U(\tilde{x}_i, \tilde{x}_{-i})] = E[U(E[\tilde{x}_i], \tilde{x}_{-i} - \pi_{-i,i})]$$ (5)

We can also define it more specifically in the class of AMU functions

$$E[u_i(\tilde{x}_i)] + E[u_{-i}(\tilde{x}_{-i})] = u_i(E[\tilde{x}_i]) + E[u_{-i}(\tilde{x}_{-i} - \pi_{-i,i})].$$ (6)

### 2.2 Increased risk aversion and ordinal preferences

We first remark on a fundamental property of risk aversion coefficients regarding a single attribute in multivariate utility functions in general and in AMU functions in particular, by considering how they relate to the ordinal preference ordering. In general, it is not possible to vary the risk aversion regarding a single attribute while keeping the risk aversion coefficients regarding other attributes and ordinal preferences fixed. For the additive case, we state that two different AMU functions can never be compared in their risk preferences independently of their ordinal preference ordering.

Specifically we state:

**Proposition 1.** Assume two twice continuously differentiable utility functions in two attributes $U(x_1, x_2)$ and $V(x_1, x_2)$ with $r^V_i(x_i, x_{-i}) \neq r^U_i(x_i, x_{-i})$ for some $i \in \{1, 2\}$ and at least one pair $x_i, x_{-i}$. Then

1. if $r^V_{-i}(x_i, x_{-i}) = r^U_{-i}(x_i, x_{-i})$ at the same pair $x_i, x_{-i}$, $U(x_1, x_2)$ and $V(x_1, x_2)$ do not imply the same ordinal preference ordering.

2. if $U(x_1, x_2)$ and $V(x_1, x_2)$ are AMU functions, $U(x_1, x_2)$ and $V(x_1, x_2)$ do not imply the same ordinal preference ordering.

The extension of all statement to more than one attribute and their proofs are provided in Appendix A. Though the proposition is stated in terms of changes in Arrow-Pratt risk aversion, it also holds for changes in Ross risk aversion. This follows because every change in Ross risk aversion is also a change in Arrow-Pratt risk aversion.
The proposition is not new to the literature, even though it is not usually stated explicitly. In his analysis of labor supply behavior and risk aversion, Chetty (2006) implicitly relies on the first item of Proposition 1. He uses ordinal preferences over labor supply and income to determine restrictions on risk aversion over money. Our proposition states the general theoretical result underlying this approach. A further implication is that no such thing as a comparative static towards the coefficient of risk aversion regarding a single attribute can exist. Any change in this parameter will either change the risk aversion coefficients regarding other attributes, the ordinal preferences between attributes, or both.

The second item considers the special case of AMU functions. Among others, Attanasio and Weber (1989) have made the observation that for additive and homogeneous intertemporal utility functions, the elasticity of substitution and the coefficient of relative risk aversion are reciprocals of one another. More broadly, the literature on stationary and recursive intertemporal preferences (originated by Koopmans, 1960) has shown that ordinal preferences and risk aversion are jointly determined for stationary and recursive AMU functions (Bommier et al., 2017). However, as has been shown by Epstein (1983), stationarity only appears in those AMU functions which evaluate each attribute with the same component utility function. It can be alluded that an analogous result holds for AMU functions with differing component utility functions. Nevertheless, we state the result explicitly here and provide its proof in the appendix.

The immediate consequence of the proposition for the context of our study is that AMU functions can never be compared solely in terms of risk aversion, even if the risk aversion coefficients towards all attributes are changed at the same time. Any non-linear, additivity preserving transformation of the utility function will change more aspects of the preference functional than just the risk aversion coefficients. It is, however, common to apply such transformations in the literature. Life-cycle consumption models, for example when analyzing optimal financial holdings over time, often use a utility function which is additively separable over time and report robustness checks regarding the parameter determining risk aversion in the assumed parametric form of the component utility functions. As is already discussed above, such comparisons not only reflect the implications of a change in risk aversion but reflect changes in other aspects of the preference functional as well (Richard, 1975; Epstein and Zin, 1989). For true comparative statics in risk aversion, one would have to consider concavification of the entire preference functionals (Kihlstrom and Mirman, 1974).
However, such a transformation does not preserve additivity and cannot address the situation in which one wants to change risk aversion regarding a single attribute only. We cover the implications of this in the next section.

2.3 Nonmonotonicity of cross-attribute risk premiums

We exemplify the implications of Proposition 1 for risk premiums using a general setting. In what follows, we will relate the coefficient of risk aversion regarding attribute $i$ to the risk premiums $\pi_{j,i}$, $j \in \{i, -i\}$, to see which results from the univariate case carry over to the multivariate case. Initially, it seems very appealing to simply apply a concave transformation as in Pratt (1964) to the component function of an AMU function. The appeal lies in the ease of application. Pratt transformations offer a simple way of deriving comparative statics and thus facilitating calibrations. The omnipresent iso-elastic utility function, for example, uses exactly such transformations by varying the coefficient of relative risk aversion through adjusting its open parameter. However, even though we know from Keeney (1973) that such a transformation will increase the risk aversion coefficient of the utility function regarding the attribute of the component function, we also know from Proposition 1 that this risk aversion coefficient cannot be interpreted as in the univariate case.

We begin with the on-attribute risk premium. Consider two multiattribute utility functions $U(x_1, x_2)$ and $V(x_1, x_2)$ for which for some $i$ it holds that $r^V_i (x_i, x_{-i}) \geq r^U_i (x_i, x_{-i}) \forall x_i, x_{-i}$. Let both $\tilde{x}_1$ and $\tilde{x}_2$ be random variables. For utility function $U(\cdot)$, the on-attribute risk premium for attribute $i$ is then defined as $E[u_i(\tilde{x}_i)] + E[u_{-i}(\tilde{x}_{-i})] = u_i(E[\tilde{x}_i] - \pi^U_{i,i}) + E[u_{-i}(\tilde{x}_{-i})]$ or simply $E[u_i(\tilde{x}_i)] = u_i(E[\tilde{x}_i] - \pi^U_{i,i})$. Since this definition corresponds to the definition of the risk premium in the univariate case, we can conclude from Pratt (1964) that in the additive multivariate case $r^V_i (x_i, x_{-i}) \geq r^U_i (x_i, x_{-i}) \forall x_i, x_{-i}$ also implies $\pi^V_{i,i} \geq \pi^U_{i,i}$. Due to the relationship between the two orderings, this also holds if $v_i(x_i)$ is more Ross risk averse than $u_i(x_i)$.

This convenient monotonicity of the on-attribute risk premium does, however, not carry over to the cross-attribute risk premium.\footnote{See also Karni (1979, footnote 2) for a result with a similar intuition. He, however, only varies the weights of the component utility functions, not their curvature.} Specifically, it is possible that $\pi^V_{-i,i} < \pi^U_{-i,i}$ even though the component utility function $v_i(x_i)$ exhibits higher Arrow Pratt risk aversion than $u_i(x_i)$.
To show this, consider a component utility function \( v_i(x_i) = b[\lambda v_i(x_i) - h(x_i)] \) with \( b > 0 \). \( v_i(x_i) \) is a positive affine transformation of a Ross transformation of \( u_i(x_i) \). Therefore, \( v_i(x_i) \) is more Ross risk averse than \( u_i(x_i) \). Recall that every Ross transformation is also a Pratt transformation. In consequence, \( v_i(x_i) \) is also more Arrow Pratt risk averse than \( u_i(x_i) \).

Now rearrange (6) such that

\[
E[u_i(\tilde{x}_i)] + E[u_{-i}(\tilde{x}_{-i})] - u_i(E[\tilde{x}_i]) = E[u_{-i}(\tilde{x}_{-i} - \pi^{U,i}_{-i,i})]
\]  

(7)

and define the new cross-attribute risk premium, \( \pi^{V,i}_{-i,i} \), via

\[
E[v_i(\tilde{x}_i)] + E[u_{-i}(\tilde{x}_{-i})] - v_i(E[\tilde{x}_i]) = E[u_{-i}(\tilde{x}_{-i} - \pi^{V,i}_{-i,i})].
\]  

(8)

Due to \( u'_{-i} > 0 \), we can evaluate the expression \( F = E[u_{-i}(\tilde{x}_{-i} - \pi^{U,i}_{-i,i})] - E[u_{-i}(\tilde{x}_{-i} - \pi^{V,i}_{-i,i})] \) to determine the sign of \( \pi^{V,i}_{-i,i} - \pi^{U,i}_{-i,i} \):

\[
F = [E[u_i(\tilde{x}_i)] + E[u_{-i}(\tilde{x}_{-i})] - u_i(E[\tilde{x}_i])] - [E[v_i(\tilde{x}_i)] + E[u_{-i}(\tilde{x}_{-i})] - v_i(E[\tilde{x}_i])]
\]  

(9)

Using the definition of \( v_i(x_i) \) gives

\[
F = [E[u_i(\tilde{x}_i)] + E[u_{-i}(\tilde{x}_{-i})] - u_i(E[\tilde{x}_i])] - [E[u_{-i}(\tilde{x}_{-i})] + b \lambda [E[u_i(\tilde{x}_i)] - u_i(E[\tilde{x}_i])] + bE[h(\tilde{x}_i)] - h(E[\tilde{x}_i])].
\]  

(10)

We rearrange and evaluate

\[
F = (1 - b \lambda) [E[u_i(\tilde{x}_i)] - u_i(E[\tilde{x}_i])] + b [h(E[\tilde{x}_i]) - E[h(\tilde{x}_i)]].
\]  

(11)

Taking the limit of this expression for \( b \) approaching zero renders

\[
\lim_{b \to 0} F = E[u_i(\tilde{x}_i)] - u_i(E[\tilde{x}_i]).
\]  

(12)

Thus, since the sign of \( F \) is the sign of \( \pi^{V,i}_{-i,i} - \pi^{U,i}_{-i,i} \), we know that for \( u''_i < 0 \) and sufficiently small \( b, \pi^{V,i}_{-i,i} < \pi^{U,i}_{-i,i} \) can appear. This is counterintuitive. The mechanic behind this result, however, is simple. We can write out the preference functional including the Ross transformation on the
component function for attribute 1 as:

\[ V(x_1, x_2) = b[\lambda u_1(x_1) + h(x_1)] + u_2(x_2) \]  

(13)

This makes it apparent that a decrease in \( b \) increases the weight of \( x_2 \) in comparison to \( x_1 \). As such even though any risk in \( x_1 \) decreases the utility more for \( V(\cdot) \) than for \( U(\cdot) \), the compensation of \( x_2 \) needed for such a risk can decrease since the importance of \( x_1 \) decreases with decreasing \( b \).

### 2.4 Contextual additive multivariate utility functions

In this section, we introduce the class of contextual additive multivariate utility (CAMU) functions which uses a construction similar to contextual utility developed by Wilcox (2011) for stochastic applications of univariate utility functions. We define the context of a decision problem as the minimum and maximum achievable values for each attribute, respectively denoted \( x_{i \text{min}} \) and \( x_{i \text{max}} \).

The contextual component utility function is then defined as

\[ \bar{u}_i(x_i) = u_i(x_i) - u_i(x_{i \text{min}}) \]

\[ u_i(x_{i \text{max}}) - u_i(x_{i \text{min}}) \]

whereas \( u_i(\cdot) \) can be any twice continuously differentiable utility function with \( u_i' > 0 \).\(^7\) A CAMU function in two attributes is defined as follows (for arbitrary positive constants \( b_1 \) and \( b_2 \)):

\[ \bar{U}(x_1, x_2) = b_1 \bar{u}_1(x_1) + b_2 \bar{u}_2(x_2) \]  

(14)

While the class of CAMU functions is more restrictive than the class of AMU functions, we can actually show that, given a context, for any function in the latter class, there is a function in the former class which represents the same preferences. To see this, take an arbitrary AMU function \( U(x_1, x_2) = u_1(x_1) + u_2(x_2) \) as defined above. The corresponding contextual function is given by (14) when setting \( b_1 = u_1(x_1^{\text{max}}) - u_1(x_1^{\text{min}}) \) and \( b_2 = u_2(x_2^{\text{max}}) - u_2(x_2^{\text{min}}) \). The function \( \bar{U}(x_1, x_2) \) is now a positive affine transformation of \( U(x_1, x_2) \) and thus implies the same preferences.

We can increase the risk aversion coefficient regarding a single attribute in these contextual functions by applying a positive concave transformation \( g(\cdot) \) to the attribute’s contextual component utility function. The resulting new multivariate utility function will again be contextual if the

\(^7\)It is apparent that each contextual component utility function is bounded.

\(^8\)The function would be fully specified with one constant multiplied with one of the component utility functions, because the overall function \( \bar{U}(x_1, x_2) \) is again invariant to positive affine transformations. We maintain the structure with two weighting constants because we want to be explicit about the weights and do not want to raise confusion with the case of \( n \) attributes discussed in Appendix A.
transformation \( g(\cdot) \) itself is a contextual function on the context \([0, 1]\). Wilcox (2011) shows that
\[
\bar{v}_i(x_i) > \bar{u}_i(x_i) \quad \text{for all values of } x_i \in [x_i^{\min}, x_i^{\max}] \text{ if } \bar{v}_i(x_i) = g(\bar{u}_i(x_i)).
\]
This is a convenient property for all changes in risk aversion. In the univariate case it implies the decision-maker to become "stochastically more risk averse" in the sense that if lottery \( L_1 \) is a mean preserving spread of lottery \( L_2 \), then in a stochastic expected utility model (as, e.g., discussed in Debreu, 1958) the probability of a decision-maker with utility function \( \bar{v}(\cdot) \) to choose \( L_2 \) over \( L_1 \) is strictly higher than that of a decision-maker with utility function \( \bar{u}(\cdot) \). For independent mean preserving spreads in individual attributes, this property carries over to AMU functions.\(^9\)

For our later results and for applications to behavioral decision models, comparisons of the derivatives of \( \bar{v}_i(x_i) \) and \( \bar{u}_i(x_i) \) will be interesting. The following lemma, which is due to Liu and Wang (2017)\(^10\), states that for increases in Ross risk aversion, the second derivatives can be compared.\(^11\)

**Lemma 1.** For two twice continuously differentiable, increasing and strictly concave contextual component utility functions \( \bar{u}_i(x_i) \) and \( \bar{v}_i(x_i) \) such that \( \bar{v}_i(x_i) \) is more Ross risk averse than \( \bar{u}_i(x_i) \), the function \( f(x_i) = \bar{v}_i(x_i) - \bar{u}_i(x_i) \) is concave for all values of \( x_i \in [x_i^{\min}, x_i^{\max}] \).

The first immediate consequence of the Lemma in combination with the result from Wilcox (2011) is that the difference in slopes of the two component utility functions \( \bar{u}_i(x_i) \) and \( \bar{v}_i(x_i) \) is neither universally positive nor universally negative. To see this, consider the definition of the contextual component utility functions. It implies that \( \bar{u}_i(x_i^{\min}) = \bar{v}_i(x_i^{\min}) = 0 \) and \( \bar{u}_i(x_i^{\max}) = \bar{v}_i(x_i^{\max}) = 1 \). The difference between the two functions, \( f(x_i) \) is thus zero at the end points of the context, and positive and concave in between. As such, the slope of \( f(x_i) \) must first be positive and then negative.

The main use of Lemma 1, however, is the following. With it, we can show that the class of CAMU functions has the convenient property that for any attribute \( i \) both on-attribute risk premiums and cross-attribute risk premiums are, ceteris paribus, monotonically increasing with an

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\(^9\)Since in AMU functions, the absolute difference in expected utilities is the sum of the absolute differences in the expected component utilities, this property follows immediately from the proof in Wilcox (2011).

\(^10\)Even though a result similar to Lemma 1 is shown in Liu and Wang (2017), we still provide the proof in the appendix for completeness.

\(^11\)An increase in Arrow-Pratt risk aversion is not sufficient for the second derivatives to be comparable. Take the two functions \( \bar{u}_i(x_i) = x_i^{0.5} \) and \( \bar{v}_i(x_i) = x_i^{0.05} \). Both are contextual utility functions on the context \([0, 1]\) and \( \bar{v}_i(x_i) \) is more Arrow-Pratt risk averse than \( \bar{u}_i(x_i) \), but the function \( f(x_i) = \bar{v}_i(x_i) - \bar{u}_i(x_i) \) is not concave on \([0, 1]\). We thank Marciano Siniscalchi for pointing out this helpful counterexample.
increase in Ross risk aversion regarding attribute $i$. This is summarized in the following theorem:

**Theorem 1.** For two twice continuously differentiable, contextual additive multivariate utility functions $\bar{U}(x_1, x_2) = b_1 \bar{u}_1(x_1) + b_2 \bar{u}_2(x_2)$ and $\bar{V}(x_1, x_2) = b_1 \bar{v}_1(x_1) + b_2 \bar{v}_2(x_2)$ with $\bar{U}^1, \bar{U}^2 > 0; \bar{U}^{11} < 0$ and $\bar{v}_1(x_1)$ being weakly more Ross risk averse than $\bar{u}_1(x_1)$, the following properties hold:

1. $\pi^{\bar{V}}_{1,1} \geq \pi^{\bar{U}}_{1,1} \forall x_j \in [x_j^{\min}, x_j^{\max}], j \in \{i, -i\}$
2. $\pi^{\bar{V}}_{2,1} \geq \pi^{\bar{U}}_{2,1} \forall x_j \in [x_j^{\min}, x_j^{\max}], j \in \{i, -i\}$

Theorem 1 and the discussion of CAMU functions above shows how useful the contextual construction can be. By using models with contextual utility functions, the applications will be limited to models which can accommodate bounded utility functions and increases in Ross risk aversion. However, since any bounded AMU function has a contextual counterpart implying the same preferences, there are no other limitations due to their application.

### 2.5 Changes in ordinal preferences

Kihlstrom and Mirman (1974) demonstrate that two agents with different ordinal preferences cannot solely be compared on the basis of their risk aversion. Let the first agent prefer outcome A over outcome B and the second agent prefer outcome B over outcome A. The first agent will then prefer outcome A over a gamble of outcome A and outcome B. The second agent will prefer the gamble over the certain outcome of A. At the same time, the second agent will prefer a certain outcome B over a gamble of outcomes A and B, while the first agent has reversed preferences. As such, it is difficult to compare the two agents with respect to their risk preferences.

Proposition 1 establishes that such an example can be constructed for every two AMU functions which differ in their risk aversion at at least one point. The answer of Kihlstrom and Mirman (1974) to that dilemma is not to compare such functions at all. However, this limits comparability of multivariate utility functions to a small number of special cases and makes using AMU functions impossible. Using the contextual utility functions still changes ordinal preferences, but does so in a way that risk premiums change consistently. It will further be shown that ordinal preferences are also changed in a sensible way.
We use an illustrative setting to show how preferences change. Consider the following CAMU function: \( \bar{U}(x_1, x_2) = b_1 \bar{u}_1(x_1) + b_2 \bar{u}_2(x_2) \). The agent is distributing a budget \( B \) between two attributes \( x_1 \) and \( x_2 \) at unit prices.\(^{12}\) This leads to a natural context \([0, B]\) for both \( \bar{u}_1(\cdot) \) and \( \bar{u}_2(\cdot) \).

The solution to this problem can be described by the first order condition:

\[
  b_1 \bar{u}_1'(x_1^* \bar{U}) - b_2 \bar{u}_2'(B - x_1^* \bar{U}) = 0 \tag{15}
\]

The question then is how the solution changes when the risk aversion of one of the component utility functions is changed. Without loss of generality, we focus on attribute \( x_1 \). We construct \( \bar{V}(x_1, x_2) = b_1 \bar{v}_1(x_1) + b_2 \bar{u}_2(x_2) \) with \( \bar{v}_1(x_1) \) being more Ross risk averse than \( \bar{u}_1(x_1) \). To determine the change in the optimal amount of attribute \( x_1 \), we evaluate the first order condition using the alternative preference function at the point \( x_1^* \bar{V} \):

\[
  \frac{\partial \bar{V}}{\partial x_1} \bigg|_{x_1^* \bar{V}} = b_1 \bar{v}_1'(x_1^* \bar{V}) - b_2 \bar{u}_2'(B - x_1^* \bar{V}) \tag{16}
\]

Knowing the second order condition being fulfilled and substituting (15) we can see that \( x_1^* \bar{V} > (=, <) x_1^* \bar{U} \) is equivalent to \( \bar{v}_1'(x_1^* \bar{U}) - \bar{u}_1'(x_1^* \bar{U}) > (=, <) 0 \).

It would be counterintuitive if the solution was trivial such that increased risk aversion regarding one attribute would always lead to more or less preference for that attribute. This would immediately give rise to the possibility of constructing decision situations such as the one discussed in the introduction. Rather, the solution should change in a non-trivial, but systematic way in the form of a trade-off. Increased risk aversion regarding an attribute makes the agent averse to very low allocations of the good. At the same time, the marginal utility decreases more rapidly due to the increased risk aversion and thus the curve flattens out for high values. The intuitive way to structure the trade-off should thus be as follows: if the initial value of \( x_1^* \bar{U} \) was low, the new allocation should be higher. On the other hand, if the initial value of \( x_1^* \bar{U} \) was high, the new allocation should be lower. An AMU function with contextual component utility functions fulfills exactly this structure. As can be seen from the discussion of Lemma 1, \( f'(x_1^{min}) = \bar{v}_1'(x_1^{min}) - \bar{u}_1'(x_1^{min}) > 0 \) and

\(^{12}\) Alternatively, the problem can be thought of as centralized resource allocation involving a utilitarian-type welfare function. The feasible set of allocations need not be defined just by the budget line and could also be restricted by the two agent’s minimal acceptable levels (i.e. a core of a cooperative game). We do not model this here for sake of simplicity of exposition, but it is clear that the above formulation does not affect generality.
\[ f'(x_1^{max}) = \tilde{v}'_1(x_1^{max}) - \tilde{u}'_1(x_1^{max}) < 0. \] Since the function \( f(x_1) \) is single peaked, the comparative static changes sign only once.

We consider next how the point at which this change in sign takes place behaves. We denote this point, for which \( f'(x_1^{x \in V}) = 0 \), as \( x_1^0 \). Obviously, \( x_1^0 \) depends on the exact change in the component function from \( \tilde{u}(\cdot) \) to \( \tilde{v}(\cdot) \), a change which can take various forms. Here, we consider one specific scenario based on a simple subclass of Ross transformations. Namely, we focus on those functions which can be described by \( v_k(x) = \lambda u(x) + kh(x) \) with \( h(x) > 0, -\lambda u'(x) < h'(x) \leq 0, h''(x) \leq 0 \) and \( 0 \leq k < -h'(x) u'(x) \) for all \( x \in [x_{min}, x_{max}] \). We show in Appendix B that this class of functions is ordered such that Ross risk aversion is increasing in \( k \). Thus, we can exemplify how \( x_1^0 \) changes in the increase in Ross risk aversion, by considering its change in \( k \). \( x_1^0 \) is implicitly defined through

\[ f'(x_1^0) = \left( \frac{\lambda u'(x_1^0)}{\lambda[u(x_1^{max}) - u(x_1^{min})]} + \frac{kh'(x_1^0)}{h(x_1^{max}) - h(x_1^{min})} \right) - \frac{u'(x_1^0)}{u(x_1^{max}) - u(x_1^{min})} = 0. \] (17)

From the implicit function theorem, we know that
\[ \frac{\partial x_1^0}{\partial k} = -\left( \frac{\partial f'}{\partial x_1} \right)^{-1}. \] From \( k < -\frac{u'(x)}{h'(x)} \) and \(-\lambda u'(x) < h'(x)\), we know that \( kh'(x) > -\lambda u'(x) \) which implies \( k[h(x_1^{max}) - h(x_1^{min})] > -\lambda u'(x) \left( h(x_1^{max}) - h(x_1^{min}) \right) \). Thus, \( \frac{\partial f'}{\partial x_1} = \frac{\lambda u'(x_1^0) + \lambda u''(x_1^0)}{\lambda[u(x_1^{max}) - u(x_1^{min})] + k[h(x_1^{max}) - h(x_1^{min})]} - \frac{h''(x_1^0)}{\lambda[u(x_1^{max}) - u(x_1^{min})] + k[h(x_1^{max}) - h(x_1^{min})]} \) < 0 which implies the sign of \( \frac{\partial x_1^0}{\partial k} \) to be equal to that of \( \frac{\partial f'}{\partial k} \).

From \( \frac{\partial f'}{\partial k} = \left( \frac{\lambda[u(x_1^{max}) - u(x_1^{min})] + k[h(x_1^{max}) - h(x_1^{min})]}{\lambda[u(x_1^{max}) - u(x_1^{min})] + k[h(x_1^{max}) - h(x_1^{min})]} \right)^{-1} < 0 \), we can hence deduce \( \frac{\partial x_1^0}{\partial k} < 0 \).

The point at which the comparative statics changes sign thus decreases in the increase of Ross risk aversion in the class of functions considered above. As stated before, marginal utility at low values of the attribute is increased with increased risk aversion. This is paired with lower marginal utility for high allocations of the attribute. Put differently: the agent is strongly averse to losing some of that attribute if he has little and, at the same time, is not very sensitive to losing some of the attribute if he has a lot of it. This can be related to the situation of insurance demand. A risk averse decision-maker will purchase insurance even at a higher loading and thus willingly give up money in a state without a loss if that protects him from very low wealth levels in the loss state. With more risk aversion regarding one of the attributes, both of these effects get stronger: the agent gets particularly averse to losses at very low levels of the attribute and less sensitive to losses at higher levels. Thus, the point \( x_1^0 \) must decrease in the increase in risk aversion.
3 Application: Calibration and risk premiums in the Köszegi-Rabin Model

The model on reference dependent risk attitudes introduced by Köszegi and Rabin (2006, 2007, henceforth KR) signifies a major step in the literature on decisions under risk as it constitutes a coherent way of endogenizing the reference point into the decision model. The model has shown to be very useful in different applications, such as compensation schemes (Herweg et al., 2010) or insurance demand (Sydnor, 2010). The way in which the model allows for reference points to endogenously arise from expectations also finds strong support in the empirical literature (e.g. Sprenger, 2015). Additionally, the model defines an interaction between large scale consumption utility and small stakes gain-loss utility. It bases this interaction on prior empirical observations and KR (2007) generate a set of exemplary results which reflect intuition sensibly.

Given wealth \( w \) and a reference point \( \rho \), KR represent preferences in their model as follows:

\[
u(w|\rho) = m(w) + \mu(m(w) - m(\rho)).\tag{18}\]

Here, \( m(\cdot) \) is the consumption utility function and \( \mu(\cdot) \) is the gain-loss utility function. In KR’s initial set of axioms A0-A4, the consumption utility function \( m(w) \) is assumed to be concave in \( w \), while the gain-loss utility function is assumed to be concave over gains (when \( w \geq \rho \)), to be convex over losses (when \( w < \rho \)) and to feature loss aversion.

The model is part of a class of modern behavioral economics models which assume that utility is derived from more than one attribute.\(^{13}\) In it, wealth leads to both reference independent consumption utility and reference dependent gain-loss utility. Even though wealth is the root of both consumption utility and gain-loss utility, they have to be seen as separate attributes. While a certain amount of wealth always leads to the same consumption utility in every decision situation, the same amount can, depending on the decision situation, lead to positive, negative or neutral gain-loss utility. The preference functional is thus a multiattribute utility function in which the first attribute is wealth and the second attribute is the divergence of consumption utility this wealth implies compared to a reference level of consumption utility. The functional form of the KR model

\(^{13}\)As mentioned in the introduction and is discussed in Köszegi and Rabin (2007), other models in behavioral economics feature similar constructions. Particularly regret aversion theories (such as the one introduced by Loomes and Sugden, 1982) are very similar and indeed can benefit from our results in a similar manner as the KR model discussed here.
as it is recited in equation (18) further implies that the utility function is additive between the two attributes and thus represents an AMU functional with component utility functions $m(\cdot)$ and $\mu(\cdot)$.

Risk preferences are determined jointly by both functions.

In this section, we will discuss and partially solve an issue in this model that was already noted by KR (2006; 2007, footnotes 6 and 12, respectively). Namely, due to the additive nature of the KR model, the shapes of its component functions alone are not informative regarding risk aversion and thus risk premiums (in terms of changes in wealth $w$) of the model. This makes the calibration of the full model a tedious process in which the function for small stakes gain-loss utility has to be elicited pointwise over the entire context (see Appendix B in Kőszegi and Rabin, 2007). We show generally that contextual component utility functions and increases in Ross risk aversion are sufficient conditions such that the risk premiums for surprise gains or losses are monotonic in the risk aversion of the small stakes gain-loss utility function. This makes the shape of the function more informative and allows for an easier calibration of the model. We then give an example for a parametric class of component utility functions in which a single parameter determines both the risk aversion of the small stakes gain-loss utility function and the risk premium over surprise gains or losses.

We consider here a situation in which the decision-maker is evaluating a surprise gain or loss with his initial wealth as the reference point. In such situations, loss aversion does not influence the decision-maker’s preferences. In its original form, the model is an AMU function with attributes large scale consumption and consumption deviation from the reference point. In this form, changing the curvature of one of the component functions does not monotonically shift the risk aversion of the preference functional. To see this, consider the coefficient of absolute risk aversion of the KR preference functional as it is given in equation (18) for surprise gains.\footnote{The argument for a surprise loss is analogous.} To avoid confusion through long equations, we drop the arguments of the functions.

\[
x^{m,\mu}(w + x|w) = -\frac{u''(w + x|w)}{u'(w + x|w)} = -\frac{m'' + m'\mu' + (m')^2\mu''}{m' + m'\mu'}
\]  

(19)

To show that in general the risk aversion of the entire preference functional is not directly related to the risk aversion of either $m(\cdot)$ or $\mu(\cdot)$, we use a Pratt (1964) transformation to form
\[ \nu(\cdot) = g(\mu(\cdot)) \]. We then compare the new risk aversion coefficient with the one in equation (19), by evaluating

\[ r^{m,\nu}(w + x|w) - r^{m,\mu}(w + x|w) = \frac{m'' + m''\mu' + (m')^2\mu''}{m' + m'\mu'} - \frac{m'' + m''\nu' + (m')^2\nu''}{m' + m'\nu'}. \] (20)

The last expression does not have a clear interpretation, since \( \mu'' - \nu'' \) does not have a clear sign. Changing \( \mu(\cdot) \) changes the additive composition of \( \mu(\cdot) \) and \( m(\cdot) \). As such, making one of the functions more risk averse can make the overall functional less risk averse, such that the risk premium for a given risk is decreased.

The argument with regard to changes in the risk aversion of \( m(\cdot) \) is corresponding to the argument for \( \mu(\cdot) \) given above. Again we can consider the change in the risk aversion coefficient of \( u(w + x|w) \) if \( m(\cdot) \) is changed to \( n(\cdot) = g(m(\cdot)) \):

\[ r^{n,\mu}(w + x|w) - r^{m,\mu}(w + x|w) = \frac{-g''(m')^2(1 + \mu') + \mu''n'(m' - n')}{n'(1 + \mu')}. \] (21)

The sign of the difference can again not be determined uniquely. \( m' - n' \) does not have a definitive sign. By Theorem 1 in Pratt (1964) this also implies that risk premiums do not change monotonically in the Arrow-Pratt risk aversion coefficient of either component utility function \( m(\cdot) \) or \( \mu(\cdot) \).

As already noted by KR, the property highlighted in Equations (20) and (21) makes the component utility functions difficult to interpret. For \( m(\cdot) \) this is expected and also not problematic. The structure of the KR preference functional explicitly allows for the forces of consumption utility risk aversion and small stakes risk aversion over changes in consumption utility to counteract one another. Since consumption utility is present in both terms, changing the curvature of this function is not intended to have a monotonic effect. Additionally, the function for consumption utility appears in an isolated fashion in some equilibria of the KR model. Its risk aversion coefficient thus has an intuitive and meaningful interpretation in those equilibria and the comparative statics there are as reported in Pratt (1964). The same argument cannot be made for the function \( \mu(\cdot) \). Firstly, it is never present in isolation and as such never has an intuitive and meaningful interpretation by
itself. Secondly, and more importantly, it is only present in one of the two additive terms of the overall functional and thus its curvature should have monotonic implications.

Panel (a) of Figure 2 offers a numeric example of how risk premiums can develop non-monotonically in the risk aversion of $\mu(\cdot)$. We consider an individual with preferences as in the example of KR: $u(w|\rho) = 10,000 \log(w) + \mu(10,000 \log(w) - 10,000 \log(\rho))$ with $\mu(x) = x^\alpha$ for $x \geq 0$ and $\mu(x) = -3(-x)^\alpha$ for $x < 0$. We focus on $\mu(\cdot)$ for which $\alpha$ is inversely related to its Arrow-Pratt risk aversion. We now consider different surprise gambles from a reference point equal to the starting wealth of $1,000,000$. As can be seen, the risk premiums for accepting a wager to win either $5,000$ or $10,000$ do not change monotonically in the risk aversion of $\mu(\cdot)$. Both first increase but later decrease in $\alpha$, even though the risk aversion of function $\mu(\cdot)$ monotonically decreases in $\alpha$. This shows that the influence of risk aversion in $\mu(\cdot)$ on the risk premium of a gamble in the gain domain is not unique in sign.

![Figure 2](image_url)

(a) Risk premiums of the original KR preference functional. Risk premiums do not change monotonically in the risk aversion of $\mu(\cdot)$ which is inversely related to $\alpha$.

(b) Risk premium of the modified preference functional using a contextual component utility function for $\mu(\cdot)$ where Arrow-Pratt risk aversion and Ross risk aversion are positively related to $\beta$.

Figure 2: Panels display the risk premium for a 50% surprise chance of winning different amounts with the reference point equal to the initial wealth of $1,000,000$ for different parameter values in the chosen functional form of $\mu(\cdot)$.

The general result in equation (20) and its numerical example in panel (a) of Figure 2 show that any sensitivity analysis regarding the risk aversion of $\mu(\cdot)$ in the KR model is difficult to interpret. This makes the calibration of the overall preference functional so difficult as there is no such concept.

\footnote{The functional form of $\mu(\cdot)$ for negative values is only stated for the sake of completeness and has no influence on the results shown in Figure 2.}
as a risk premium which can be linked to the cardinal structure of $\mu(\cdot)$. However, our Theorem 1 directly implies sufficient conditions under which the risk premium in the model is monotonic in the risk aversion of the function $\mu(\cdot)$. Using a contextual component utility function for $\mu(\cdot)$ and applying increases in Ross risk aversion rather than just in Arrow-Pratt risk aversion leads to this monotonicity and makes $\mu(\cdot)$ more interpretable. If these two changes are applied the sign of $r^{m,\gamma}(w + x|w) - r^{m,\mu}(w + x|w)$ becomes positive (due to $\bar{\mu}'' - \bar{\nu}'' > 0$, cf. Lemma 1) as would be expected and an analysis of changes in the risk aversion coefficient becomes more intuitive.\footnote{As mentioned above, an increase in Ross risk aversion always increases the Arrow-Pratt risk aversion as well. Thus, applying shifts in Ross risk aversion allows for a numerical analysis of shifts in risk aversion.}

In behavioral economics in general and for the calibration of preference functionals in particular, it is often important to assume a parametric structure of the preference functional. For this, it can be desirable to have a single parameter that controls the risk aversion of a component utility function. For the applicability of our theoretical results, it is thus important that there are parametric functions in which a single parameter is monotonically related to the Ross risk aversion of the function. Ross (1981) himself introduces one such class which is given by:

$$u_i(x_i) = x_i - \beta e^{-x_i}$$  \hspace{1cm} (22)

Here, Ross risk aversion (and thus also Arrow-Pratt risk aversion) is monotonically increasing in $\beta$. Another such class is the class of quadratic utility functions ($u_i(x_i) = x_i - \gamma x_i^2$). However, this class has very low empirical support (Stott, 2006). We thus recommend working with the functional form given by equation (22).

An application of this function is shown in panel (b) of Figure 2 which exemplifies our general result. We use the same situation as displayed in panel (a) but use a contextual component utility function instead. We choose the context $[0, \bar{b}(\log(1,010,000) - \log(1,000,000))]$ for $\bar{\mu}(x)$ for $x \geq 0$ and $[\bar{b}(\log(1,000,000) - \log(1,010,000)), 0]$ for $\bar{\mu}(x)$ for $x < 0$. The preference functional now is $u(w|\rho) = \bar{b}\log(w) + \bar{\mu}(\log(w) - \bar{b}\log(\rho))$ with $\bar{\mu}(x) = \frac{x - \beta e^{-x} + \frac{\beta}{\bar{\mu}'(x_{\text{min}})}}{\bar{\mu}'(x_{\text{max}}) - \beta e^{-x_{\text{max}}} + \beta}$.\footnote{Note that $\mu(x_{\text{min}}) = -\beta$ from equation (22) by construction since $x_{\text{min}} = 0$.} It can be seen how the proposed functional leads to monotonic changes of the risk premium in the risk aversion of the gain-loss utility function.
Using a contextual utility function and increases in Ross risk aversion can, however, only ameliorate the comparative statics issues, not eliminate them all-together. Ordinal preferences still shift to a certain extent when the risk aversion of a contextual component utility function is changed. This happens in a more consistent and intuitive manner than before, but it can never be fully eliminated without leaving the the additive paradigm.

4 Discussion and conclusions

In this paper, we aim to change the risk aversion regarding a single attribute of an AMU function. We start by remarking that this is impossible to do without changing either the ordinal preference structure or the additivity of the utility function. Kihlstrom and Mirman (1974) argue that only utility functions with the same ordinal preferences should be compared based on their risk aversion. However, this leaves only few functions eligible for comparison and, amongst other things, makes AMU functions incomparable regarding their risk aversion. This in turn implies that many of the comparisons of AMU functions, such as those done in calibration exercises of additive behavioral models, cannot be made unconditionally.

Following Wilcox (2011), we introduce the class of CAMU functions which are a subset of AMU functions. In this class, changing the Ross risk aversion regarding one attribute leads to monotonic changes of both the risk premium in that attribute and the risk premium in all other attributes. Additionally, while applying a Ross risk aversion increasing transformation to a component utility function does not only change the risk aversion, but also affects the ordinal preference ordering, it does so in an intuitive way. The class of functions lends itself to making popular behavioral economics preference functionals more interpretable and can significantly simplify their calibration as we exemplify in the popular decision model by Kőszegi and Rabin (2006, 2007). The functions, however, do not fully remedy the problems. When applying Theorem 1, one always needs to be aware that it is not only the risk aversion that changes. Thus, some of the sensitivity analysis results can, in part, also stem from the change in ordinal preferences.

It is generally the case that as soon as one makes structural assumptions for a multiatribute utility function, it is possible that situations exist in which ”undesirable” behavior is implied by the assumed form. This is true for AMU functions which have been subject to several criticisms in the
theoretical literature (e.g. Richard, 1975; Epstein and Zin, 1989). However, structural assumptions are often necessary, because certain questions, such as welfare analyses or measurements of the value of the statistical life, cannot be analyzed without them. When structural assumptions are made, AMU functions are still one of the most commonly used forms. In canonical economic analyses, the additive form for intertemporal decisions is either chosen due to its prevalence (e.g. Hall and Jones, 2007) or to simplify calculations which is often necessary for computationally exhaustive calibrations (e.g. Einav et al., 2010). In behavioral applications, it is by far the most prevalent form to model two attributes such as own wealth and that of others (Fehr and Schmidt, 1999; Trautmann, 2009) or to model two stimuli that both result from monetary payments (e.g. Loomes and Sugden, 1982; Kőszegi and Rabin, 2006, 2007). Studies in these fields thus rely on additivity assumptions despite the well-known criticisms of them. Indeed, even though it is known that changing risk aversion in the additive intertemporal model also changes the intertemporal rate of substitution, robustness checks on the coefficient of risk aversion of the component functions abound. Similarly, studies in behavioral economics bestow meaning on the coefficients of their component functions even though such meaning is also never separable from ordinal preferences.

Showing what statements can and cannot be made within the setting of AMU functions is an important question which we address in this paper. We aim at allowing researchers that want to stay in the additive paradigm to have component utility functions with more meaningfully interpretable coefficients. As we have shown in our application this greatly simplifies the calibration of preference functionals as cross-attribute risk premiums are now sufficient for the exercise whilst before the entire preference functional had to be elicited point-wise over the relevant context (see Kőszegi and Rabin, 2007, Appendix B).

A different approach to comparing the risk aversion of two AMU functions has been proposed by Jindapon and Neilson (2007). In their discussion of higher-order generalizations of risk aversion measures, they derive a comparative static on a prevention problem by equating the marginal utility of $u_i$ and $v_i$ at the expectation of the risk on attribute $x_i$. While they aim at demonstrating a different point with their analysis, they briefly discuss the comparability of the two agents and argue against it because ordinal preferences have changed. Their argument is in line with our warning that ordinal preferences will always be affected by a transformation of a component utility function and that this should always be kept in mind when deriving comparative statics. The advantage
of our approach over that of Jindapon and Neilson (2007) is simply that their transformation is
dependent on the expected value of the specific risk studied. It can thus not be applied as flexibly as
our approach. In particular, their approach can likely not be applied to the calibration of preference
functionals, because such methods commonly change not only the riskiness but also the expectation
of lotteries (as, e.g., in Holt and Laury, 2002). Despite its higher degree of flexibility, our approach
is still dependent on the context of the decision problem. This also does not make it universally
applicable, but it is applicable to any risk which is bounded within the context, irrespective of
its expected value. The note of Liu and Wang (2017), which extends the paper by Jindapon and
Neilson (2007) and was developed independently from our work, uses a normalization which is
equivalent to our class of CAMU functions. Their first proposition then shows a result which is
related to our Theorem 1. The major difference between their result and ours is that they consider
a maximization problem which is concerning the optimal amount of reduction in risk, while we
consider the willingness to pay for the elimination of a specified amount of risk. The two concepts
are related, but the answer to one does not imply an answer to the other (Jaspersen, 2016).

Our paper clarifies what analyzing a change in risk aversion of a component utility function can
and cannot show. By introducing CAMU functions and highlighting increases in Ross risk aversion,
we provide a way of changing the concavity of component utility functions while maintaining certain
monotonicity properties. Such an analysis does change both the ordinal preference ordering and
the risk aversion coefficient regarding the argument of the respective function. However, we know
from Proposition 1 that only changing the risk aversion coefficient and maintaining both ordinal
preferences and the additive structure is, in fact, impossible.
Appendix

A Extensions to more than two attributes and proofs

A.1 Extension of Proposition 1 and proof

We can define the AMU function in \( n \) attributes as \( U(x) = U(x_1, ..., x_n) = \sum_{i=1}^{n} u_i(x_i) \) and its contextual counterpart as \( \tilde{U}(x) = \bar{U}(x_1, ..., x_n) = \sum_{i=1}^{n} b_i \tilde{u}_i(x_i) \). We now state the general version of Proposition 1 for \( n \) attributes.

**Proposition 2.** Assume two twice continuously differentiable utility functions in \( n \) attributes \( U(x_1, ..., x_n) \) and \( V(x_1, ..., x_n) \) with \( n \geq 2 \) and \( r_i^V(x) \neq r_i^U(x) \) for some \( i \) and at least one value of \( x \). Then

1. if \( r_j^V(x) = r_j^U(x) \) for some \( j \neq i \) at the same value of \( x \), \( U(x_1, ..., x_n) \) and \( V(x_1, ..., x_n) \) do not imply the same ordinal preference ordering.

2. if \( U(x_1, ..., x_n) \) and \( V(x_1, ..., x_n) \) are AMU functions, \( U(x_1, ..., x_n) \) and \( V(x_1, ..., x_n) \) do not imply the same ordinal preference ordering.

**Proof.** We start with the first item, which we prove by contradiction. Assume that the functions \( U(\cdot) \) and \( V(\cdot) \) represent the same ordinal preferences over \( n \) attributes. This implies that there exists a non-decreasing and thus monotonic function \( \Gamma(\cdot) : \mathbb{R} \to \mathbb{R} \) such that \( \Gamma(U) = V \). Assume further that \( r_i^U(x) \neq r_i^V(x) \) at some value of \( x \). From the definition of \( V \), we know \( r_i^V(x) = -\frac{\partial^2 V}{\partial x_i^2} = -\frac{\partial^2 V}{\partial x_i^2} \left( \frac{\partial V}{\partial x_i} \right)^2 + \frac{\partial V}{\partial x_i} \frac{\partial^2 U}{\partial x_i^2} \) at some value of \( x \). Our regularity conditions dictate \( \frac{\partial U}{\partial x_i} > 0 \) and \( \frac{\partial V}{\partial x_i} > 0 \) such that \( r_i^U(x) \neq r_i^V(x) \) at some value of \( x \). By the same logic as above, we see that \( r_j^V(x) = -\frac{\partial^2 V}{\partial x_j^2} \frac{\partial U}{\partial x_j} + r_j^U(x) \). So \( r_j^V(x) = r_j^U(x) \) at \( x \) for some \( j \neq i \) implies \( \frac{\partial^2 \Gamma(U(x))}{\partial U(x)^2} = 0 \) at \( x \).

This is a contradiction.

The second item is again shown by contradiction.\(^{18}\) Again assume that the AMU functions \( U(\cdot) \) and \( V(\cdot) \) represent the same ordinal preferences over \( n \) attributes, implying that there exists a non-decreasing and thus monotonic function \( \Gamma(\cdot) : \mathbb{R} \to \mathbb{R} \) such that \( \Gamma(U) = V \). Assume further

\(^{18}\)A previous version of the paper included a proof which did not require differentiability of \( U(\cdot) \) and \( V(\cdot) \). It was pointed out to us that assuming differentiability significantly reduces complexity of the proof and reflects the setting of our paper better.
that $r^U_i(x) \neq r^V_i(x)$ at some value of $x$. From above we know that $r^U_i(x) \neq r^V_i(x)$ for some value $x$ implies $\frac{\partial^2 \Gamma(U(x))}{\partial U(x)^2} \neq 0$ at $x$. From both both $U(\cdot)$ and $V(\cdot)$ being AMU functions, we know for all $j \neq i$ it holds that $\frac{\partial^2 U(x)}{\partial x_i \partial x_j} = \frac{\partial^2 V(x)}{\partial U(x)^2} \frac{\partial U(x)}{\partial U(x)} \frac{\partial U(x)}{\partial U(x)} + \frac{\partial \Gamma(U(x))}{\partial U(x)} \frac{\partial^2 U(x)}{\partial U(x)^2} \frac{\partial^2 U(x)}{\partial U(x)^2} = 0$ and thus $\frac{\partial^2 \Gamma(U(x))}{\partial U(x)^2} = 0$. This is a contradiction. 

Since Proposition 1 is a special case of Proposition 2, it is sufficient to proof the latter for showing the former.

### A.2 Proof of Lemma 1

This is a recollection of the argument by Liu and Wang (2017) and is stated here for completeness.

**Proof.** Because $\bar{u}_i(x_i)$ is more Ross risk averse than $\bar{u}_i(x_i)$, it holds that $v_i(x_i)$ is more Ross risk averse than $u_i(x_i)$. Thus, $\frac{v''(x_i^a)}{u''(x_i^a)} \geq \lambda \geq \frac{v''(x_i^b)}{u''(x_i^b)}$ for all $x_i^a, x_i^b \in [x_i^{min}, x_i^{max}]$. This implies $v''(x_i^a) \geq -\lambda u''(x_i^b)$ and $v''(x_i^b) \leq \lambda u''(x_i^a)$ for all $x_i^a, x_i^b \in [x_i^{min}, x_i^{max}]$. From the latter it follows that $v_i(x_i^{max}) - v_i(x_i^{min}) \leq \lambda [u_i(x_i^{max}) - u_i(x_i^{min})]$. Substituting the inequality of the second derivatives renders $\frac{-v''(x_i^a)}{v_i(x_i^{max}) - v_i(x_i^{min})} \geq \frac{-u''(x_i^a)}{u_i(x_i^{max}) - u_i(x_i^{min})}$ and thus $-\bar{v}''(x_i) \geq -\bar{u}''(x_i)$ or $\bar{v}''(x_i) - \bar{u}''(x_i) \leq 0$ for all $x_i \in [x_i^{min}, x_i^{max}]$. 

### A.3 Extension of Theorem 1 and proof

We start by defining the relevant concepts for the case of $n$ attributes. By virtue of the additive functional form, we may, without loss of generality, assume attributes $x_1$ through $x_j$ to be the on-attributes and attributes $x_i$ for $i > j$ to be the cross-attributes. Consistent with the notation in Section 2.1, we define the on-attributes as $x_{1,\ldots,j}$ and the cross-attributes as $x_{j+1,\ldots,n}$. We define the on-attribute and cross-attribute risk premiums in many attributes in correspondence with Paroush (1975). We define the on-attribute risk premium in multiple attributes as a vector

$$(\pi_{1,\ldots,j}, \ldots, \pi_{j,\ldots,n}) \in \Pi_{1,\ldots,j;1,\ldots,j}$$

which is a solution to

$$E[U(\tilde{x}_1, \ldots, \tilde{x}_j, \tilde{x}_{j+1}, \ldots, \tilde{x}_n)] = E[U(E[\tilde{x}_1] - \pi_{1,\ldots,j}, \ldots, E[\tilde{x}_j] - \pi_{j,\ldots,j}, \tilde{x}_{j+1}, \ldots, \tilde{x}_n)]. \quad (23)$$
Similarly, we define the cross-attribute risk premium in multiple attributes as a vector 
\[ (\pi_{j+1;1}, \ldots, j, \ldots, \pi_{n;1}, \ldots, j) \in \Pi_{j+1, \ldots, n;1, \ldots, j} \] 
which is a solution to 
\[
E[U(\tilde{x}_1, \ldots, \tilde{x}_j, \tilde{x}_{j+1}, \ldots, \tilde{x}_n)] = E[U(E[\tilde{x}_1], \ldots, E[\tilde{x}_j], \tilde{x}_{j+1} - \pi_{j+1;1}, \ldots, j, \ldots, \tilde{x}_n - \pi_{n;1}, \ldots, j)].
\] (24)

Theorem 1 can in some ways be extended to multiple attributes but not in every case. We will 
cover all possible cases and prove those which can be generalized. For the theorem to be generaliz-
able, we first need a notion of how two risk premiums in multiple attributes can be compared. For 
either equation (23) and equation (24) the solution vector is not unique. To make the risk premiums 
nevertheless comparable, we adopt the definition by Paroush (1975) and, for the on-attribute risk 
premium, define \( \Pi^1_{1, \ldots, j;1, \ldots, j} \) to be larger than \( \Pi^2_{1, \ldots, j;1, \ldots, j} \), in short \( \Pi^1 > \Pi^2 \) if \( \pi^1_{i,1, \ldots, j} > \pi^2_{i,1, \ldots, j} \) 
whenever for all other \( k \neq i, \pi^1_{k,1, \ldots, j} = \pi^2_{k,1, \ldots, j} \). The cross-attribute risk premiums are compared in 
the same way.

If the risk premium is paid in more than one attribute, Theorem 1 can only be extended to those 
cases in which the risk premium is not paid in any of the attributes for which the component utility 
function is changed. The reason is that changing the risk aversion in the component utility function 
of an attribute not only changes the risk premium on that attribute, but also the ordinal preferences 
between the attributes. Assume, for example, \( n = 2 \). If the risk in attribute 1 is supposed to be 
reduced, the component utility function of attribute 1 is changed and if the risk premium is paid in 
both attributes, then not only the risk aversion of each attribute, but also the ordinal preferences 
between both attributes play a role. Since the ordinal preferences do not change monotonously 
(as was shown in Section 2.5), there is always the possibility that their change influences the risk 
premium in the opposite direction to the influence of the concavification.

It could be argued that this posits a trade-off. If the change in ordinal preferences was monotonous, 
one could always devise a counterintuitive decision situation as the one given in the introduction. 
However, a non-monotonous change leads to a restriction on the generalization of Theorem 1. This 
trade-off is, however, an artifact of the case in which only one utility function is changed. If both util-
ity functions are changed, even a monotonous change in ordinal preferences due to a concavification 
(which could, e.g., be achieved by applying a transformation \( g(\cdot) \) on all component utility functions 
which is restricted such that \( g'(\cdot) < 1 \)) would not be sufficient. Since the component utility functions
differed before, even applying the same transformation to all of them would render changes to ordinal preferences which depend on the original component utility functions. This would prohibit any general statement regarding the change in the risk premium. We also know from Propositions 1 and 2 that such a situation will appear under any transformation which preserves additivity.

These considerations leave three cases for which Theorem 1 can be generalized: firstly, if the risk in multiple attributes is reduced and the cross-attribute risk premium is paid out of a single attribute, secondly, if the risk in one or more attributes is reduced and the cross-attribute risk premium is paid out of more than one attribute and thirdly (and trivially) to the on-attribute risk premium if risk in only one attribute is reduced while the utility function has more than one other attribute. This is summarized in Theorem 2 below.

**Theorem 2.** For two twice continuously differentiable, contextual additive multivariate utility functions $\bar{U}(x_1, ..., x_n) = \sum_{i=1}^{n} b_i \bar{u}_i(x_i)$ and $\bar{V}(x_1, ..., x_n) = \sum_{i=1}^{j} b_i \bar{v}_i(x_i) + \sum_{i=j+1}^{n} b_i \bar{u}_i(x_i)$ with $\bar{U}_i > 0 \; \forall \; i$, $\bar{U}_{ij} < 0 \; \forall \; i \leq j < n$ and each $\bar{v}_i(x_i)$ being weakly more Ross risk averse than the corresponding $\bar{u}_i(x_i)$, the following properties hold:

1. $\Pi^{\bar{V}}_{ij} \geq \Pi^{\bar{U}}_{ij}, \forall \; x \in [x_{\text{min}}, x_{\text{max}}]$ and $i \leq j$.

2. $\Pi^{\bar{V}}_{j+1, ..., h, ..., k} \geq \Pi^{\bar{U}}_{j+1, ..., h, ..., k}, \forall \; x \in [x_{\text{min}}, x_{\text{max}}]$ and $1 \leq h \leq k, j$.

**Proof.** The first property follows from the additivity of the utility functions and from the fact that the set of contextual additive multivariate utility functions is a subset of the additive multivariate utility functions.

To prove the second property, we use the additive nature of the utility function and define the risk premiums for both utility functions.

$$\sum_{i} b_i E[\bar{u}_i(\bar{x}_i)] = \sum_{i \in [1, h]} b_i E[\bar{v}_i(\bar{x}_i)] + \sum_{i \in [h, j]} b_i \bar{v}_i(E[\bar{x}_i]) + \sum_{i \in [j, k]} b_i \bar{u}_i(E[\bar{x}_i] - \pi^{\bar{U}}_{i;h, ..., k})$$

$$+ \sum_{i \in [k, n]} b_i E[\bar{u}_i(\bar{x}_i) - \pi^{\bar{U}}_{i;h, ..., k}]$$

(25)

$$\sum_{i \in [1, j]} b_i E[\bar{v}_i(\bar{x}_i)] + \sum_{i \in [j, n]} b_i E[\bar{u}_i(\bar{x}_i)] = \sum_{i \in [1, h]} b_i E[\bar{v}_i(\bar{x}_i)] + \sum_{i \in [h, j]} b_i \bar{v}_i(E[\bar{x}_i] - \pi^{\bar{V}}_{i;h, ..., k})$$

$$+ \sum_{i \in [h, j]} b_i \bar{v}_i(E[\bar{x}_i]) + \sum_{i \in [k, n]} b_i E[\bar{u}_i(\bar{x}_i) - \pi^{\bar{V}}_{i;h, ..., k}]$$

(26)

Subtracting equation (26) from equation (25) and simplifying renders:
\[0 = \sum_{i \in [h,j]} b_i \left[ \mathbb{E} \left[ \bar{u}_i(\tilde{x}_i) \right] - \tilde{u}_i(\mathbb{E}[\tilde{x}_i]) \right] + \left( \mathbb{E}[\tilde{v}_i(\tilde{x}_i)] - \bar{v}_i(\mathbb{E}[\tilde{x}_i]) \right) - \sum_{i \in [j,k]} b_i \left[ \tilde{u}_i(\mathbb{E}[\tilde{x}_i] - \pi_{i;h,,...,k}^{U} - \tilde{u}_i(\mathbb{E}[\tilde{x}_i] - \pi_{i;h,,...,k}^{V}) \right] \]

(27)

We now consider the elements of the first sum of the right hand side. As in Lemma 1, we define

\[f(x_i) = \tilde{v}_i(x_i) - \tilde{u}_i(x_i).\]

Using Lemma 1, we can then state:

\[
\begin{align*}
&f(\mathbb{E}[\tilde{x}_i]) - \mathbb{E}[f(\tilde{x}_i)] \geq 0 \\
&\tilde{u}_i(\mathbb{E}[\tilde{x}_i]) - \tilde{u}_i(\mathbb{E}[\tilde{x}_i]) - \mathbb{E}[\tilde{v}_i(\tilde{x}_i)] - \mathbb{E}[\tilde{v}_i(\tilde{x}_i)] \geq 0 \\
&\mathbb{E}[\tilde{u}_i(\tilde{x}_i)] - \tilde{u}_i(\mathbb{E}[\tilde{x}_i]) - \mathbb{E}[\tilde{v}_i(\tilde{x}_i)] - \mathbb{E}[\tilde{v}_i(\tilde{x}_i)] \geq 0
\end{align*}
\]

(28)

Thus, the other two sums of the right hand side taken together must be positive. Applying the comparison concept for multi-element risk premiums by Paroush (1975), we need to show that for an arbitrary \(l > j\) it holds that \(\pi_{i;h,,...,k}^{U} = \pi_{i;h,,...,k}^{V} \forall i \neq l\) implies \(\pi_{l;h,,...,k}^{U} \leq \pi_{l;h,,...,k}^{V}\). This can be seen by

\[
\pi_{i;h,,...,k}^{U} = \pi_{i;h,,...,k}^{V} \forall i \neq l \Rightarrow \begin{cases} 
\tilde{u}_l(\mathbb{E}[\tilde{x}_l] - \pi_{l;h,,...,k}^{U}) \geq \tilde{u}_l(\mathbb{E}[\tilde{x}_l] - \pi_{l;h,,...,k}^{V}) & \text{for } l \leq k \\
\mathbb{E}[\tilde{u}_l(\tilde{x}_l - \pi_{l;h,,...,k}^{U}) - \tilde{u}_l(\tilde{x}_l - \pi_{l;h,,...,k}^{V})] \geq 0 & \text{for } l > k
\end{cases}
\]

(29)

Since Theorem 1 is a special case of Theorem 2, it is sufficient to proof the latter for showing the former.

While the possibilities to extend Theorem 1 are limited, they do offer the generalization to the possibly most relevant case of many attributes at risk. Emerging challenges like cyber risk lead to risk in several dimensions such as reputation, intellectual property and interruption of operations. Investing into cyber security infrastructure decreases these risk at the cost of monetary expenses. It is very fathomable that certain events, such as a big data leak at a competitor, make managers more risk averse regarding some of those dimensions. Theorem 2 offers a tool for calibration and economic analyses of such situations.
Given a utility function \( u(x) \), the functions \( v_k(x) \) are defined by the transformation \( v_k(x) = \lambda u(x) + kh(x) \) with \( \lambda > 0 \), \( h(x) > 0 \), \(-\lambda u'(x) < h'(x) \leq 0 \), \( h''(x) \leq 0 \) and \( 0 \leq k < -\frac{\lambda u'(x)}{h'(x)} \) for all \( x \in [x_{\text{min}}, x_{\text{max}}] \). Since Ross risk aversion is equal for all positive affine transformations of a utility function, it does not change from \( u(x) \) to \( v_k(x) \) if \( k = 0 \). To show that Ross risk aversion is increasing in \( k \), we show that \( k_1 < k_2 \), it holds that \( \frac{v_{k_2}'(x_a)}{v_{k_1}'(x_a)} \geq \frac{v_{k_2}'(x_b)}{v_{k_1}'(x_b)} \). The inequality is equivalent to \( \frac{\lambda u''(x_a) + k_2 h''(x_a)}{\lambda u''(x_a) + k_1 h''(x_a)} \geq \frac{\lambda u''(x_b) + k_2 h''(x_b)}{\lambda u''(x_b) + k_1 h''(x_b)} \). It can be seen that it is always fulfilled, because the left-hand-side is greater 1, while the right-hand-side is smaller 1.
References


